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Exact eigenfunctions for a quantised map

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Abstract. Exact eigenfunctions for quantised linear maps on a torus are constructed. A connection with the number of periodic orbits of the classical map is established. It is shown that in the semiclassical limit the eigenfunctions become more and more random, i.e. their correlation function approaches a δ function.

1. Introduction

There has been increasing effort recently in identifying the quantum analogue of chaotic behaviour in dynamical systems (see e.g. Zaslavsky 1981, Chirikov *et al* 1981, Berry 1983). Just as in the classical case, maps have proven to be very interesting objects of study (Casati *et al* 1979, Berry *et al* 1979, Chirikov *et al* 1981, Hannay and Berry 1980, Shepelyansky 1983, Dorizzi *et al* 1984). An important result of these investigations is that quantum mechanics may impose limitations to irregular behaviour, e.g. to the effectiveness of diffusive spreading of expectational values (see also Shuryak 1976, Casati *et al* 1984). As has been shown by Fishman *et al* (1982) and Grempel *et al* (1984), this may be understood in terms of an associated 1D tight-binding model. The quasi-energy enters in the diagonal part of the tight-binding Hamiltonian and the kicking potential determines the off-diagonal part, in particular the range of the interaction. It is generally believed that for short-range interactions already quasi-periodicity in the diagonal elements is sufficient for localisation. This immediately explains the resonance structure seen in the kicked rotator (see the work on maps cited above and Izrailev and Shepelyanskii (1979)). Outside resonance, the quasi-eigenstates are localised, thus diffusion is slow, whereas at resonance they are extended and diffusion is fast. For long-range interactions, states may always be extended, so there should be no quantum threshold.

Some of the simplest models of fully chaotic behaviour in the classical limit are linear maps of the torus onto itself, a famous example being the Arnold cat-map (Arnold and Avez 1969). In the theory outlined above, they belong to the class of models with long-range interaction (because of periodicities, the kicking potential is discontinuous, so its Fourier transform falls off like $1/r$). However, because of the restriction onto a torus, they are only approximations to discontinuous linear maps on a cylinder, which is what one actually has in mind. The investigation into the quantum mechanics of these maps was begun by Hannay and Berry (1981) (henceforth referred to as HB). They treated in detail the quantisation procedure, the form of the spectrum and the Wigner function for eigenstates. Here, we would like to complete their study with a discussion of the eigenstates. In particular, we will solve the eigenvector problem for even eigenstates in closed form. This will allow us to address questions on the semiclassical behaviour of eigenvectors and to discuss localisation properties.

We begin in § 2 with a brief summary of the relevant results of HB. In § 3 we present the solution of the eigenvector problem for even states. Properties of eigenstates, and in particular their semiclassical limit, are studied in § 4. A summary of results and final comments can be found in § 5. In the appendix we collect some formulae on Gauss sums.

2. Quantising linear maps on a torus

Consider a linear map T of the phase plane (q, p) onto itself:

$$\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}. \tag{1}$$

Liouville's theorem requires T to be area-preserving, i.e. $\det T = ad - bc = 1$. To obtain a torus, we divide the plane into rectangles and identify sides. By a rescaling of coordinates, we may take the side length to be equal to 1. The map T has to leave the corner positions invariant, so $a, b, c, d \in \mathbb{Z}$.

We now summarise the results of HB.

(i) Since the wavefunctions are periodic in q, their momentum representation is discrete. But the momentum functions have to be periodic, too, so any wavefunction is discrete in both coordinate and momentum representation, with N points per period. Taking $1/N$ as units of position and momentum we may label all components by integers. Planck's constant h and N are connected via $Nh = 1$.

(ii) It turns out that this restriction of the map onto a discrete grid requires T to be one of the 'checkerboard' forms

$$T = \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}. \tag{2}$$

Examples are the families

$$T_H = \begin{pmatrix} 2m & 1 \\ 4m^2 - 1 & 2m \end{pmatrix} \qquad m \in \mathbb{Z} \tag{3}$$

and

$$T_E = \begin{pmatrix} 2\tilde{m} & 1 \\ -4\tilde{m}^2 - 1 & -2\tilde{m} \end{pmatrix} \qquad \tilde{m} \in \mathbb{Z}. \tag{4}$$

Note that the first map is elliptic for $m = 0$ and hyperbolic otherwise, whereas the second is always elliptic.

(iii) Because of the discreteness, time evolution within one period is given by a unitary $N \times N$ matrix. Generally, the matrix elements are of the form

$$U_{kl} = (\text{phase factor}) \times \exp(\mathrm{i}\pi/N) QF(k, l) \tag{5}$$

where $QF(k, l)$ denotes a quadratic form in k and l with integer coefficients and perhaps a prime factor of N as common divisor. For the two maps above, we have (for arbitrary N)
for T_H

$$U_{k,l} = (\mathrm{i}/N)^{1/2} \exp(\mathrm{i}\pi/N)(2mk^2 + 2ml^2 - 2kl) \tag{6}$$

for T_E

$$U_{k,l} = (\mathrm{i}/N)^{1/2} \exp(\mathrm{i}\pi/N)(2\tilde{m}k^2 - 2\tilde{m}l^2 - 2kl). \tag{7}$$

(iv) An important property of the quantisation procedure is the following. Let $U(T_1)$, $U(T_2)$ be the propagators for the maps T_1 and T_2. Then

$$U(T_1 \cdot T_2) = U(T_1)U(T_2). \tag{8}$$

This implies that if $T^{n(N)} = \mathrm{id} \bmod N$, i.e. if $T^{n(N)}$ acts like the identity on the torus, then $U(T)^{n(N)} = U(\mathrm{id}) = \mathbb{1}$, up to some unimportant phase factors. This implies that the eigenvalues of $U(T)$ are distributed on the unit circle with spacings given by multiples of $(2\pi/n)$. It can be shown that for elliptic maps, $n(N) = 4$ (for $N \geq 4$) whereas for hyperbolic maps, $n(N)$ is a highly erratic function of N. The following conjecture, borne out by numerical computations, will be useful below: for N an odd prime which does not divide the discriminant $D = (a+d)^2 - 4$ of T, $n(N)$ divides $N+1$ or $N-1$.

(v) Finally, we need the connection between periodic orbits of the classical map and degeneracies of eigenvalues. (Note that periodic orbits are called cycles in HB. We will reserve this term for a different, though related, object in quantum mechanics.) We begin with the observation that since the map is linear, the Wigner function will evolve in time just like a classical phase space density. Now, for eigenstates the Wigner function is invariant. This implies that it can be written as a sum over densities supported by classical periodic orbits and constant along them. Thus, eigenstates are determined by at most as many parameters as there are periodic orbits.

Consider now the most general time-independent phase space density. Classically, it is completely determined by its values along periodic orbits and quantum mechanically, it corresponds to a density matrix not containing terms mixing states with different eigenvalues. The number of free parameters in both cases has to be the same, which gives the desired relationship: the number of periodic orbits of the classical map equals the sum over eigenvalue degeneracies squared. In particular, for hyperbolic maps and for N prime and not a divisor of the discriminant, this implies that all orbits have equal length $n(N)$ (except for the fixed point at the origin). They act on $N^2 - 1 = (N-1)(N+1)$ points, so their total number is $(N^2-1)/n(N)$ (again omitting the origin).

3. Computation of eigenvectors

For convenience, we adopt the convention that all indices $j, k, l, \ldots,$ run from 0 to $N-1$, with N the period of the lattice. By periodicity all indices of unitary propagators may be taken modulo N. We then have the following symmetry in the propagators of (6) and (7):

$$U_{N-j,N-k} = U_{j,k}. \tag{9}$$

Accordingly, we may say an eigenvector transforms odd (even) under this symmetry, depending on whether it changes sign or not. We will mainly be interested in the case N prime, so even states will have $[N/2]+1$ independent components and odd states $[N/2]$, with the zeroth component fixed at zero ($[a]$ denotes the largest integer $\leq a$).

To motivate our ansatz for the eigenvectors, recall that in the continuous case, integration of a Gaussian function results in another one with perhaps different parameters. Similar results exist in number theory for some particular sums, also named after Gauss. The one we will need is

$$\sum_{k=0}^{N-1} \exp\left(\frac{2\mathrm{i}\pi}{N}(ak^2+ck)\right) = \sqrt{N}\left(\frac{2a}{N}\right) \exp\left(\frac{-\mathrm{i}\pi}{4}(N-1)\right) \exp\left(-\frac{2\mathrm{i}\pi}{N}(a(2a\backslash N)^2c^2)\right) \quad (10)$$

for a positive (otherwise substitute $a \to 2N-a$). Here $(2a\backslash N)$ denotes the unique integer inverse to $2a$ in the residue class modulo N, i.e. $2a(2a\backslash N)=1 \bmod N$. $(\frac{2a}{N})$ denotes a Jacobi symbol which takes on values ± 1. Further discussion may be found in the appendix.

The following ansatz then proves to be sufficient to generate all even eigenvectors:

$$\psi_k = \sum_{\mu=1}^{L} \exp\left(\frac{\mathrm{i}\pi}{N}a_\mu k^2 + \mathrm{i}\gamma_\mu\right) \quad (11)$$

(up to normalisation). The number of components L will be determined below.

Periodicity of ψ_k requires a_μ to be even, which also implies that ψ has the right symmetry. Action of U on one of the components $\psi_{k,\mu}$ results in

$$\begin{aligned}\sum_k U_{lk}\psi_{k,\mu} &= \mathrm{e}^{\mathrm{i}\gamma_\mu}\left(\frac{\mathrm{i}}{N}\right)^{1/2} \exp\left(\frac{2\pi\mathrm{i}}{N}ml^2\right)\sum_k \exp\left(\frac{\mathrm{i}\pi}{N}[(2m+a_\mu)k^2-2lk]\right)\\ &= \mathrm{e}^{\mathrm{i}\gamma_\mu}\mathrm{i}^{1/2}\left(\frac{2m+a_\mu}{N}\right)\exp\left(-\frac{\mathrm{i}\pi}{4}(N-1)\right)\\ &\quad\times\exp\left(\frac{\mathrm{i}\pi}{N}[2m-(2m+a_\mu)(2m+a_\mu\backslash N)^2]l^2\right)\\ &= \mathrm{e}^{\mathrm{i}\gamma'_\mu}\exp\left(\frac{\mathrm{i}\pi}{N}a'_\mu l^2\right).\end{aligned} \quad (12)$$

The condition for ψ to be an eigenvector reads

$$\sum_k U_{lk}\psi_k = \mathrm{e}^{\mathrm{i}\lambda}\psi_l. \quad (13)$$

Now, for our ansatz to be successful a'_μ and γ'_μ of (12) have to be contained in (11), which we can be assured of by taking the a_μ and δ_μ generated by the recursion relations

$$a_{\mu+1} = 2m-(2m+a_\mu)(2m+a_\mu\backslash N)^2 \quad (14a)$$

$$\gamma_{\mu+1} = \lambda+\gamma_\mu+\pi\left[\frac{1}{2}-\frac{N}{4}+\left(\frac{2m+a_\mu}{N}\right)\right]. \quad (14b)$$

By (14a) any even a_μ is uniquely mapped onto an even $a_{\mu+1}$, so there is some $L \leqslant N$ such that $a_{L+1}=a_1$. We call a sequence $a_1 \ldots a_L$ a cycle and L its length.

Since our matrix U is unitary, the eigenvectors may be chosen real. For the ansatz (11) this means that for all a_μ, $2N-a_\mu$ ($=-a_\mu \bmod 2N$) should also be part of the cycle. It was found numerically that, indeed, in some cases all the negative a_μ were part of the cycle (the so-called complete cycle). In the majority of cases, however, the negative a'_μ formed a cycle of their own, with exactly the same length and the same condition on the eigenvalues (see below). We will return to this when we discuss their length. Thus, incomplete cycles come in pairs.

Knowing the cycles a_μ, we may exploit (14*b*) to obtain the eigenvalues. Summation over all μ yields

$$L\left(\lambda+\frac{\pi}{2}-\frac{N\pi}{4}\right)+\pi\sum_{\mu=1}^{L}\binom{2m+a_\mu}{N}=0 \bmod 2\pi. \tag{15}$$

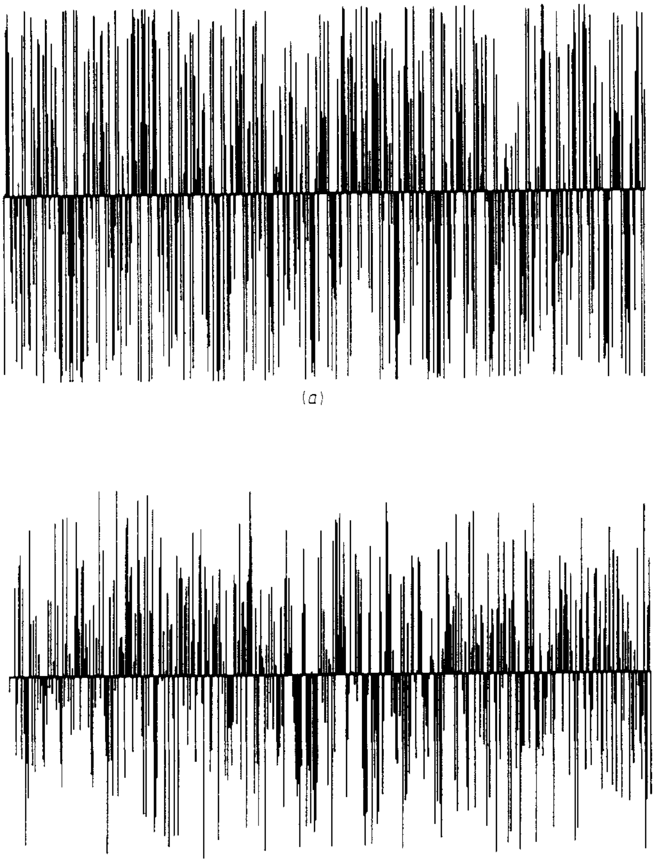
This shows that each cycle of length L yields exactly L eigenvectors.

The length of the cycles was numerically found to be $n(N)$, $n/2$ or 1. A cycle of length 1 occurs only if $n(N)$ divides $N-1$ and it then takes care of the eigenvalue with degeneracy one higher (the case of multiplicity one less always seems to be realised in the space of odd vectors). As for the long cycles, recall that they give eigenvectors for L eigenvalues, spaced at $(2\pi/L)$ (or multiples thereof). But we also know that all eigenvalues have spacing $(2\pi/n(N))$, so the length has to divide $n(N)$. The observation that L is so large may be interpreted semiclassically as follows. The projection of the Wigner function onto coordinate space yields, by construction, the wavefunction. As noted before (§ 2(v)), the Wigner function for eigenstates has as many parameters as there are periodic orbits, so the same should hold for the eigenfunctions. Our findings for the length of the cycles indicate that the periodic orbits group according to the multiplicities of eigenvalues but within each group all orbits contribute to the eigenfunction. This is in agreement with figures 4 and 5 in HB where Wigner functions of eigenstates and periodic orbits for a hyperbolic map (equation (3) with $m=1$) are compared. We have not been able to answer the question whether there is a one-to-one correspondence between classical periodic orbits and the coefficients a_μ and δ_μ.

In table 1 we list data on the number of cycles and their length for various values of N and m. What typical eigenvectors look like is shown in figure 1.

Table 1. Periods and cycles for various values of m and N. Abbreviations: n = period, C = complete cycle, IC = pairs of incomplete cycles. Numbers in parentheses give the length of the cycle.

	N			
m	9001	9007	9011	9013
1	$n=2250$ 1 × C (1125) 3 × IC (1125) 1 × IC (1)	$n=9008$ 1 × C (4504)	$n=9010$ 1 × C (4505) 1 × IC (1)	$n=9012$ 1 × C (4506) 1 × IC (1)
2	$n=1125$ 1 × C (1125) 3 × IC (1125) 1 × IC (1)	$n=3002$ 1 × C (1501) 2 × IC (1501) 1 × IC (1)	$n=9010$ 1 × C (4505) 1 × IC (1)	$n=4507$ 1 × C (4507)
3	$n=9002$ 1 × C (4501)	$n=4504$ 1 × C (2252) 1 × IC (2252)	$n=1502$ 1 × C (751) 5 × IC (751)	$n=9014$ 1 × C (4507)
4	$n=4501$ 1 × C (4501)	$n=1501$ 1 × C (1501) 2 × IC (1501) 1 × IC (1)	$n=9012$ 1 × C (4506)	$n=3004$ 1 × C (1502) 2 × IC (1502) 1 × IC (1)
5	$n=1125$ 1 × C (1125) 3 × IC (1125) 1 × IC (1)	$n=9008$ 1 × C (4504)	$n=9010$ 1 × C (4505) 1 × IC (1)	$n=9012$ 1 × C (4506) 1 × IC (1)



(a)

(b)

Figure 1. Examples of chaotic eigenstates for $N = 9011$, $m = 1$. (a) Eigenstate belonging to the incomplete 1-cycle (multiplicity two); (b) an eigenstate belonging to the complete 4505-cycle (multiplicity one). 480 elements are shown.

We have not been able to extend this ansatz to cover odd eigenvectors as well. Some reasons are indicated in the appendix. Closely related is the omission of the linear term in (11), also commented upon in the appendix.

4. Properties of eigenvectors

We now would like to discuss statistical properties of typical eigenfunctions. By typical, we mean eigenvectors belonging to eigenvalues of low multiplicity, preferably multiplicities one or two. The reason is that in highly degenerate eigenspaces linear combinations of eigenvectors may be taken to create (almost) any form desired. An extreme example of this is provided by the elliptic map (4) and its propagator (7). Since the propagator also contains a quadratic form in the exponent, the procedure of § 3 may be applied to produce eigenvectors of much the same appearance as figure 1. Yet, we know that $n(N) = 4$, so each eigenvalue is highly degenerate and may be simplified considerably. Similar things happen for hyperbolic maps with $n(N) \ll N$.

One of the striking features of the eigenvectors is their highly erratic appearance (see figure 1). One way to test this is by studying the correlation function

$$C(l)=\sum_{k=0}^{N-1}\psi_{k+l}\psi_k. \tag{16}$$

The prototype of the correlation function to be expected is obtained for $\psi_k=\cos((\pi/N)ak^2)$:

$$C(l)=\tfrac{1}{2}N\delta_{l,0}+\tfrac{1}{4}\sqrt{N}\cos((\pi/N)a'l^2+\gamma) \tag{17}$$

(again up to normalisation of ψ_k).

This clearly shows that as $N\to\infty$ the first term dominates by a factor of $\sqrt{N}$ and the ψ_k become more and more δ-correlated. Generally, ψ_k will consist of L cosine terms and $C(l)$ will have L^2 terms, L of which contain δ functions weighted with N, the remaining L^2-L being of the form $\sqrt{N}\cos((b\pi/N)k^2+\gamma)$. Because of (17) we may consider the latter as a sum of L^2-L random variables of mean 0 and variance $O(\sqrt{N})$, so their total weight will be $O(L\sqrt{N})$, again a factor $\sqrt{N}$ smaller than that of the $\delta_{l,0}$ term. In conclusion we may say that typical eigenvectors for hyperbolic maps are almost random variables.

It has been conjectured for some time that eigenstates of non-integrable systems are highly irregular functions with complicated nodal patterns, etc. Berry (1977) arrived at this conclusion by studying the (anti-)caustic structure of the Wigner function. Shapiro and Goelman (1984) introduced a path correlation function and found numerically for the stadium billiard a transition from long-range oscillations in low-lying eigenstates to rapid decay in high-lying states. The eigenfunctions presented above are the first example where a quantitative discussion of correlation properties is possible and indeed the wavefunction is found to be a δ-correlated random variable. The distribution for its values approaches (as $N\to\infty$)

$$P(x)\,dx=\pi^{-1}(1-x^2)^{-1/2}\,dx \tag{18}$$

for eigenvectors obtained from cycles of length 1 and a Gaussian density for others (compare figures 1(*a*) and (*b*)).

5. Conclusions

In the preceding sections we have constructed exact eigenfunctions of even symmetry for a family of linear maps of the torus onto itself. The form of the eigenvectors was found to be linked to the number of periodic cycles of the classical map. The eigenstates are extended, in agreement with expectations arising from the connection with tight-binding models. We studied the correlation function $C(l)$ of the eigenstates and found it to be dominated by $C(0)$. As we approach the semiclassical limit $N\to\infty$, $C(l)/C(0)\sim O(1/\sqrt{N})$, so the randomness in the eigenvectors increases.

In the classical map the eigenvalues (Lyapunov exponents) are a measure of the degree of chaos, larger exponents corresponding to more random time evolution. The quantum version seems not to be susceptible to this: the data in table 1 show that the function $n(N)$, determining the eigenvalue degeneracies, depends rather irregularly on m also, and moreover has to repeat itself after at most N values (see (6)). The correlation function $C(l)$, though monotonic in N, is insensitive to changes in m, at least for the even eigenstates discussed. Thus it appears that in this system quantum mechanics distinguishes only between elliptic and hyperbolic, showing non-generic behaviour (highly degenerate eigenvalue, simple eigenvectors) in one case and generic behaviour (equidistantly distributed eigenvalues of multiplicity 1, random eigenvectors) in the other. This suggests that the right measures of quantum chaos, at least on the level of eigenvalues and eigenfunctions, still have to be discovered.

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Appendix

In this appendix we list various properties of Legendre and Jacobi symbols and of Gauss sums relevant to our analysis (see HB, Rademacher 1964, Schroeder 1984).

The Legendre symbol $\left(\frac{a}{p}\right)$ is defined as

$$\left(\frac{a}{p}\right)=\begin{cases}+1 & \text{if there exists an } m \text{ such that } m^2=a \bmod p\\ -1 & \text{if no such } m \text{ exists}\end{cases} \tag{A1}$$

for p prime.

The Jacobi symbol is the generalisation of this to arbitrary $N=\prod_i p_i$, p_i the prime factors:

$$\left(\frac{a}{N}\right)=\prod\left(\frac{a}{p_i}\right). \tag{A2}$$

One then has the product laws

$$\left(\frac{a_1a_2}{N}\right)=\left(\frac{a_1}{N}\right)\left(\frac{a_2}{N}\right) \tag{A3}$$

$$\left(\frac{a}{N_1N_2}\right)=\left(\frac{a}{N_1}\right)\left(\frac{a}{N_2}\right) \tag{A4}$$

and the Gauss reciprocity theorem

$$\left(\frac{a}{N}\right)\left(\frac{N}{a}\right)=(-1)^{(a-1)(N-1)/4}. \tag{A5}$$

These formulae allow an efficient computation of Jacobi symbols, although for repeated use a table may be more convenient.

In their derivation of the propagator U, HB need the averaged Gauss sum

$$S_{\text{ave}}=\lim_{M\to\infty}\frac{1}{2M}\sum_{k=-M}^{M}\exp\left(\frac{\mathrm{i}\pi}{N}(ak^2+ck)\right). \tag{A6}$$

If p is the period, then obviously

$$S=\sum_{\text{period}}\exp\left(\frac{\mathrm{i}\pi}{N}(ak^2+ck)\right)=pS_{\text{ave}} \tag{A7}$$

and we can use their results. By inspection of the exponent one finds that the period is N for $(a+c)$ even and $2N$ otherwise (assuming a and N are coprime), so the adaptation of (14) in HB reads

period N:

$$\sum_{k=0}^{N-1} \exp\left(\frac{\mathrm{i}\pi}{N}(ak^2+ck)\right)$$

$$=\sqrt{N}\left(\frac{a}{N}\right)\exp\left(-\frac{\mathrm{i}\pi}{4}(N-1)\right)\exp\left[-\frac{\mathrm{i}\pi a}{N}(a\backslash N)^2\left(\frac{c}{2}\right)^2\right] \qquad \text{for } a \text{ even, } c \text{ even} \qquad (\mathrm{A}8a)$$

$$=\sqrt{N}\left(\frac{a}{N}\right)\exp\left(-\frac{\mathrm{i}\pi}{4}(N-1)\right.\ \exp\left(-\frac{\mathrm{i}\pi a}{N}(a\backslash N)^2\left(\frac{c}{2}\right)^2\right. \qquad \text{for } a \text{ odd, } c \text{ even}$$

period $2N$:

$$\sum_{k=0}^{2N-1} \exp\left(\frac{\mathrm{i}\pi}{N}(ak^2+ck)\right)$$

$$=2\sqrt{N}\left(\frac{N}{a}\right)\exp\left(\frac{\mathrm{i}\pi a}{4}\right)\exp\left[-\frac{\mathrm{i}\pi a}{N}(a\backslash N)^2\left(\frac{c}{2}\right)^2\right] \qquad \text{for } a \text{ odd, } c \text{ even} \qquad (\mathrm{A}8b)$$

$$=0 \qquad \text{for } a \text{ even, } c \text{ odd.}$$

Note that in (A8a) we may convert one case into the other by adding N to both a and c. This ambiguity explains the restriction to a_μ even in (11). Symmetry requirements then restrict c to 0 (or N). We suspect that the odd eigenvectors are hidden in (A8b). However, we would need a closed form solution for the incomplete sum over half a period, something we have not been able to derive.

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